

Basic concepts of the Simplex Method

By now, we know that most LP solvers use some variation of Dantzig's method, called the Simplex method. In these note, we explore (a) some basic concepts about why the Simplex methods works so well, and (b) the mechanics of the method, i.e. how does it work. We begin with some background.

The beauty of convexity

We will deal with shapes. We will define a shape as a set of points in space. For example, the set of all points lying on or inside a circle of radius 1, centered on the origin, define the shape of a disc. Loosely, a convex shape has the property that if we pick any two points inside the shape, and join them by a straight line, then every point on the line lies inside the shape. The figure below illustrates the idea. Notice that the shape in Figure 1c is not convex because we can find two points, **a** and **b**, that are inside the shape, but the line joining **a** and **b** has points, e.g. **c**, lying outside the shape.

[Note: it is sufficient to use the intuitive meaning of “inside” and “outside”; those who enjoy mathematics are encouraged to take a course on topology to learn more about this].

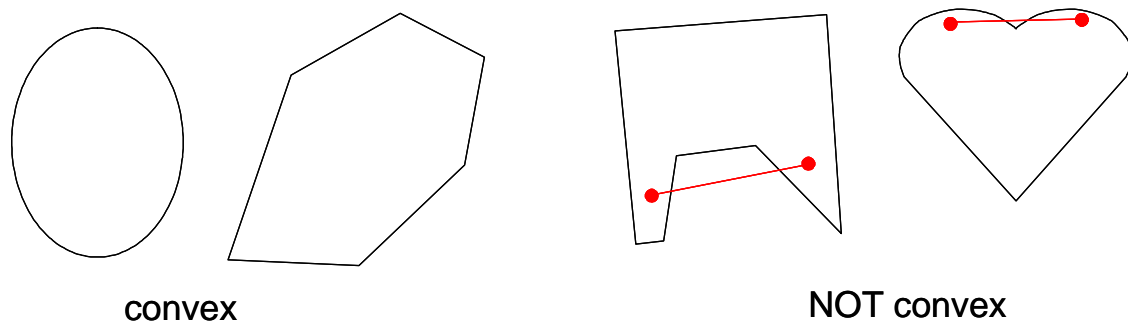


Figure 1. Examples of convex and non-convex shapes

Property 1. Convex sets have many nice properties; one that is special interest to us is that the *intersection of any two convex sets is also convex*. By extending this argument, it is easy to see that the intersection of any number of convex objects is convex.

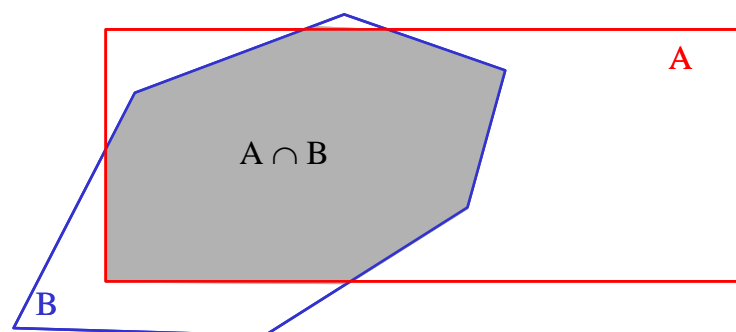


Figure 2. Intersection of convex objects is convex

In dealing with LP's, we work with linear inequalities. We now claim that the set of all points that satisfy a linear inequality form a convex set in the underlying space. For example, the inequality $ax + by \leq c$ divides the set of all points on the XY plane into two partitions. We will call such a partition a “half-plane”. You can extend this idea to 3D, with the plane $ax + by + cz \leq d$ yielding two “half-spaces”, and similarly to n-dimensions, where the half-spaces are defined by $a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b$.

The **boundary** of the half-plane is defined by the line $ax + by = c$, and in 3D, by the plane $ax + by + cz = d$, and so, in n-dimensions, by an $n-1$ dimensional shape that we call a “hyper-plane” $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$. The **boundary equation** of a constraint is obtained by replacing the inequality sign (\leq or \geq) with an equality ($=$).

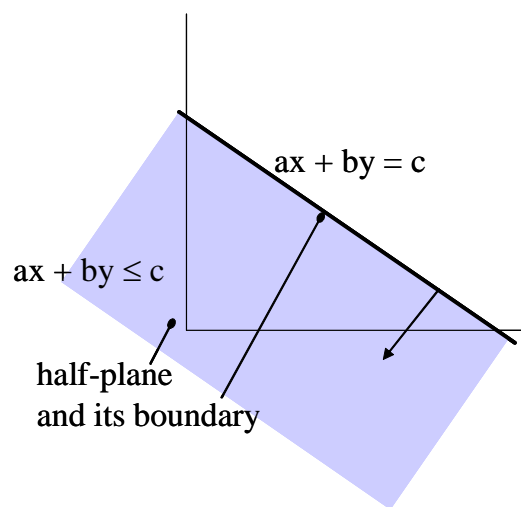


Figure 3. Half-spaces and their boundaries

Property 2. A half space is a convex shape.

We mentioned earlier that a **feasible solution** for an LP is a solution (i.e. specifying one value for each variable) that satisfies each constraint. The set of all feasible solutions of an LP, called the **feasible region**, is therefore the intersection of the half-spaces defined by each of the constraints. Now, from properties 1 and 2, we can say that the **feasible region** is a convex shape.

It is possible that:

- (i) the feasible region is empty \rightarrow there is no feasible solution.
- (ii) the feasible region is unbounded in a direction that allows the objective to increase/decrease without restriction; and
- (iii) the feasible region is bounded (e.g. in our Product mix example).

When we model a real problem and come up with the second case (the simplex method can actually recognize this case, as we shall see), it is likely that we forgot to include some constraint(s).

An optimum solution is one that maximizes (or minimizes) the objective function. In the first case, since the feasible region is empty, there is no optimum solution. Is it possible to have more than one optimum solution? As we shall see, this is possible. An interesting thing is that if there are two distinct optimum solutions, then any point on the line segment joining these two will also be an optimum solution – in other words, there will be infinite number of optimum solutions. In the 2D case, this is equivalent to the situation when one edge of the feasible region is parallel to the objective function line (i.e. the objective function and one of the constraints have the same slope).

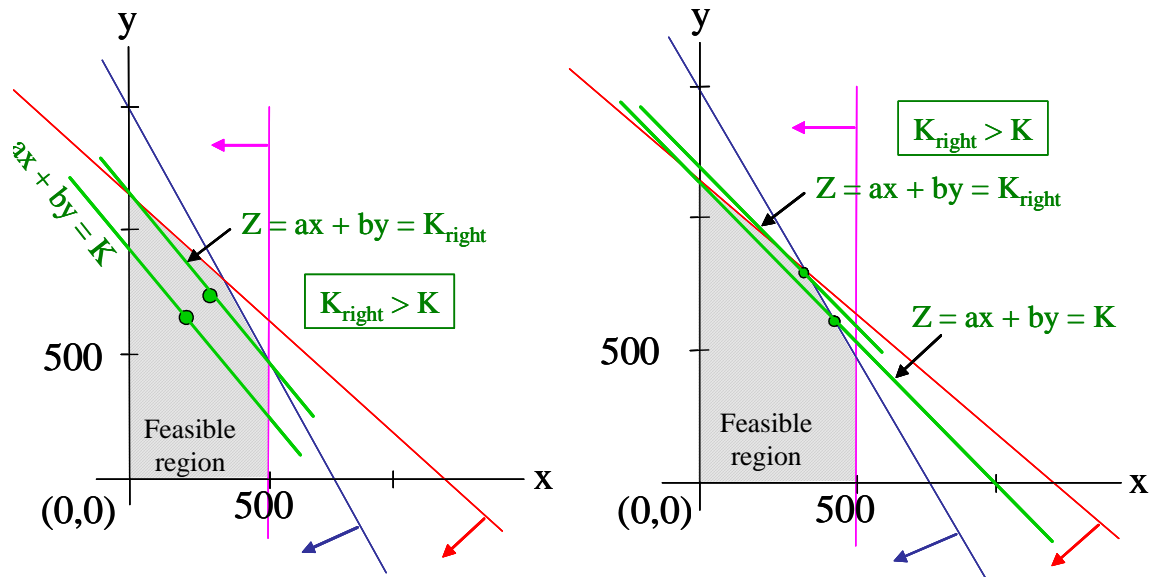
If a solution lies on the boundary of the feasible region, we call it a **boundary feasible solution**. Further, a **corner point feasible solution** is a solution that is (i) feasible, (ii) we cannot find two other feasible solution such that the corner-point solution is on the line segment joining those two. You may have noticed that the optimum solution to the Product Mix problem was a corner-point feasible solution. Further, I will urge you to re-visit our graphical arguments about the Product Mix problem to see why we should suspect that if there are one or more optimum solutions, then they must include a corner-point feasible solution.

What is a corner point? In 2D, a corner point is obtained by solving for the intersection point of two constraint boundaries (i.e. the intersection point of two lines). In 3D, to get a corner point we will need to find the intersection of three planes (i.e. three constraint boundaries). Similarly, in n dimensions, a corner point is the solution of n constraint boundary equations.

Two corner point solutions are said to be **adjacent**, if the line segment joining them is an edge of the feasible region. In fact, the key properties of corner point feasible solutions lie at the heart of the Simplex procedure, so let us list them explicitly.

Property 3. (a) If there is exactly one optimum solution, it must be at a corner point. (b) If there are multiple optimum solutions, then they must include at least two adjacent corner point solutions.

Assume that the optimum point is in the interior of the feasible region, and corresponds to an optimum value of the objective, $z = K$. Draw the hyper-plane corresponding to $z = K$. since this point is in the interior, there must be at least one point to the left, and at least one point to the right of this solution. Draw hyper-planes parallel to $z = K$ through these two points, $z = K_{\text{left}}$, and $z = K_{\text{right}}$. From our observation earlier, at least one of (K_{left} , K_{right}) must yield a better objective function value than K . For example, if the objective is to maximize z , then $K_{\text{right}} > K$. Therefore our interior point solution cannot have been optimum. Hence the optimum must lie on a corner.



(a) No optimum in interior of feasible region (b) No *single optimum point* in middle of a boundary edge

Figure 4. Geometric insight about property 4

Property 4. There are a finite number of corner point feasible solution.

Corner points lie on the boundary of the feasible region. If the feasible region is made up of constraints with inequalities, i.e. $\sum a_i x_i \leq b$, the boundary is formed by the corresponding equalities, $\sum a_i x_i = b$. Therefore a corner point is derived from solving a set of n simultaneous equations. If there are totally m constraints in our formulation, then the total number of corner points is just $C(m, n) = m!/n!(m-n)!$, which is a finite number.

Property 5. If a corner point solution is better (i.e. has a larger value of the maximization objective function) than all its adjacent corner solutions, then it must be an optimal solution.

This property is true because our feasible region is convex. Consider a corner point that is better than all its adjacent corner points, with the objective value of $z = z^*$. Draw the hyper-plane $z = z^*$. Since all the adjacent corners yield a lower objective value, the hyper-planes through each such neighbor must be on the left side of $z = z^*$. Now, if there was some other corner point that has an objective value, say $z^{**} > z^*$. Then the hyper-plane $z = z^{**}$ is on the right side of $z = z^*$. Draw a line segment joining the corner points for z^* and z^{**} . It is entirely to the right of $z = z^*$. But in the neighborhood of the corner point of z^* , the entire feasible region is to its left. Therefore, this line cannot lie entirely inside the feasible region. In other words, the feasible region is not convex. But that is impossible – so there cannot be any corner point corresponding to z^{**} . Hence z^* is the optimum.

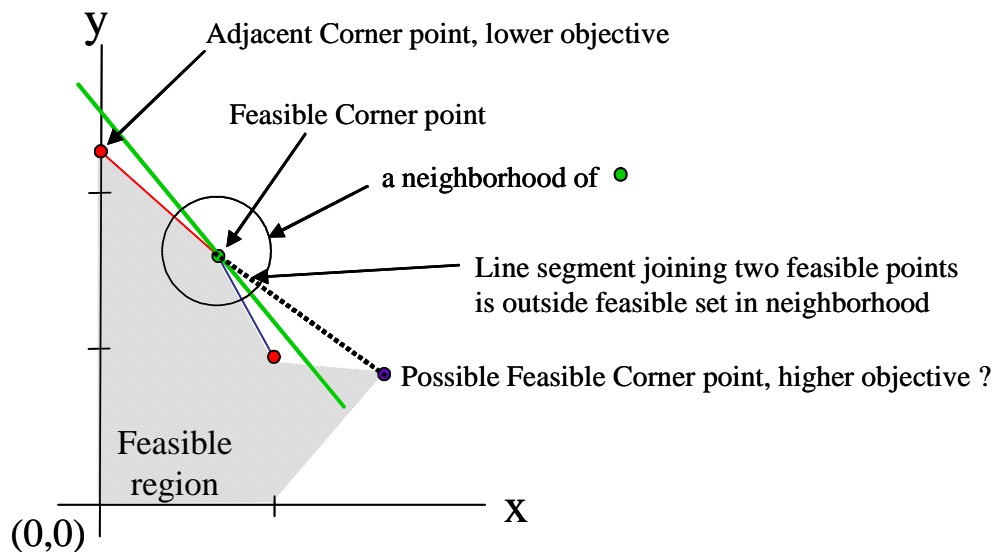


Figure 5. Graphical explanation of property 5: the beauty of convexity

The simplex method

Properties 3, 4, and 5 form the basis of the simplex method. Property 3 ensures that we only need to look at corner points. Property 4 ensures that instead of looking for the optimum over infinite number of points in the feasible set, we only need to find the best one out of a finite number of corner points. This sounds like good news, since we can just program a computer to compute each corner point, and among all of these, pick the one that maximizes the objective. This is good for small problems. But consider a medium sized LP with 50 variables and 100 constraints. There are $100!/(50! \cdot 50!) \approx 10^{29}$ corner points. Even a pretty fast computer will have a tough time listing them all (consider that if the entire earth was made of sand, then it would contain around 10^{29} grains of sand).

Fortunately, property 5 comes to our rescue – we find one corner point, and then only look at its adjacent corners; from here, we move to a neighbor that improves our objective value, and then repeat the process from that corner point. If at some stage, there is no adjacent corner point that can increase the objective value, we have found the optimum point !

To summarize, here is the Simplex method:

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1. Start at a corner point feasible solution
 2. If (there is an adjacent corner feasible point that increases the objective) then
 go to one such adjacent corner point, and repeat Step 2;
 else (report this point as the optimum point).
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What is the guarantee that this method works? Note that totally there are only a finite number of corner points; if we list the objective value at each of these points, we will get a finite sequence of numbers. Let us list them in increasing order. Simplex method begins at some point in this sequence, and at each step, it can only move to the right (increase the objective). Therefore it can only go a finite number of steps before it hits the maximum (optimum) [in other words, because it increases the objective at each step, it cannot be running in circles over the same set of solutions for infinite times].

Note that several details are missing in the above steps – how do we find a corner point solution? How do we guarantee that each corner point we examine is feasible? If several of the adjacent corner feasible points increase the objective value, which one should we select? To answer these, we must look at the algebraic details of the simplex method.

Slack variables

Note that we solve for corner points by solving a set of linear equations (by solving a set of simultaneous equations, e.g. using Gaussian elimination). In Gaussian elimination, we repeatedly “eliminate” one variable by substituting its value as a function of the remaining. For example, consider the set of equations:

$$\begin{array}{ll} x + y = 2 & [1] \\ x - 2y = 1 & [2] \end{array}$$

We “solve” this by first writing [2] as $y = (x - 1)/2$, and substituting into [1], to get: $x + (x - 1)/2 = 2$, giving $x = 5/3$. This method is equivalent to Gaussian elimination, where we just manipulate add multiples of some equation to another in order to eliminate some variable; e.g. add $2x[1] + [2]$, to get: $3x = 5$, which can be solved for x , and likewise, $[1] - 1x[2]$ gives, $3y = 1$, which yields $y = 1/3$. This method is more convenient if we have many equations and many variables.

However, in LP, we are faced with a set of inequalities, and we cannot go about adding and subtracting inequalities as we do for equations! Take for example:

$$\begin{array}{ll} x \geq 0 & [3] \\ y \geq 0 & [4] \end{array}$$

Now **WE CANNOT** conclude that: $[3] + [4]$ should yield: $x + y \geq 0$!

For example, the point $x = 3, y = -2$; it satisfies $x + y \geq 0$, but violates [4]. Thus, if we solved inequalities using the Gaussian technique, we may find a solution that is infeasible in the original set of inequalities.

In summary: (i) We want to use Gaussian elimination, (ii) Our problem has inequalities, and (iii) We cannot use Gaussian method with inequalities.

To resolve this problem, we shall convert our inequalities to equations; consider the constraint:

$$2x + y \leq 1500, \text{ where } x, y \geq 0$$

We shall write this as:

$$2x + y + s = 1500, \text{ where } x, y, s \geq 0.$$

The new variable, s , is called a slack variable. It is equally easy to convert a \geq constraint to an equality:

$$x + y \geq 200, x, y \geq 0 \text{ can be rewritten as: } x + y - s = 200, x, y, s \geq 0.$$

We are now ready to look at the algebraic form of Simplex. As you can see, each inequality constraint will introduce one extra variable, so we will be working with many variables; it is convenient to use subscripted variable names. I will use our familiar Product Mix problem as an example:

$$\begin{array}{ll} \text{maximize} & z(x, y) = 15x + 10y \\ \text{subject to} & 2x + y \leq 1500 \\ & x + y \leq 1200 \\ & x \leq 500 \\ & x \geq 0, \\ & y \geq 0 \end{array}$$

Let's introduce slack variables, and also change the variable names to, get the equivalent algebraic form, called the *standard form*:

$$\begin{array}{ll} \text{maximize} & Z = 15x_1 + 10x_2 \\ \text{subject to} & 2x_1 + x_2 + x_3 = 1500 \\ & x_1 + x_2 + x_4 = 1200 \\ & x_1 + x_5 = 500 \\ & x_i \geq 0 \text{ for all } i. \end{array}$$

I have deliberately added some spaces, to make the problem readable. Note that in general, if our original problem had m constraints and n variables, then the equality form of the problem, with the additional slack variables, will contain $(m+n)$ variables. (To be a little more precise, some constraints may have been equalities to begin with, in which case we don't need to associate a slack with them, but that has no effect on the Simplex method, as we shall see.)

So now we have m equations and $(m+n)$ variables – and from basic high-school algebra, we know that a system of m equations can be solved for at most m variables. So we do

the following: we will fix the value of n variables to be 0. Then we will be left with m equations in m unknowns, which we can solve using Gaussian elimination.

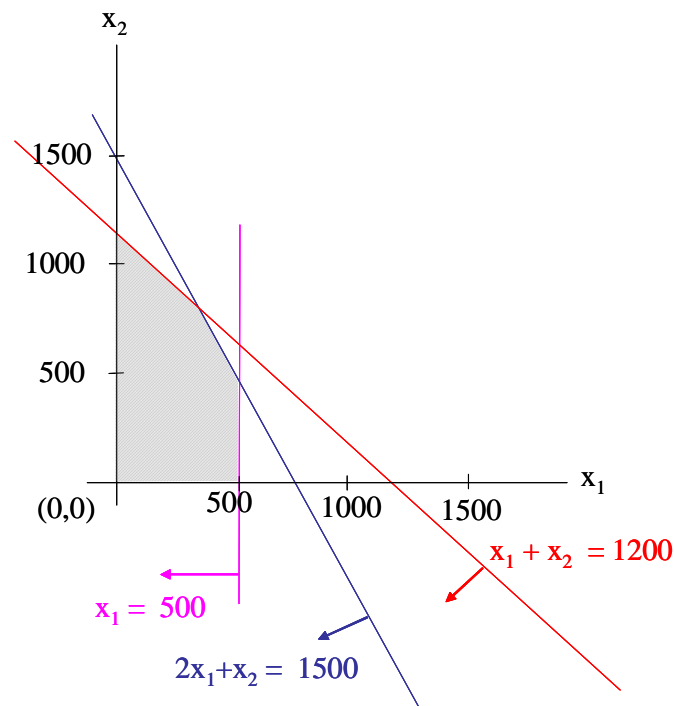


Figure 6. Graph of the product mix constraints and feasible set

Some definitions:

Let (x_1, x_2) be a solution to our original problem. Then the corresponding solution with the values of the slack variables is called an **augmented solution**. For example, the solution $(x_1, x_2) = (200, 200)$ of the original problem corresponds to the augmented solution $(x_1, x_2, x_3, x_4, x_5) = (200, 200, 900, 800, 300)$. You can get the augmented solution by easily putting the values for x_1 and x_2 in the constraint equations of the standard form.

A **basic solution** is an augmented corner-point solution. Here are two examples of basic solutions: $(500, 700, -200, 0, 0)$ is a **basic infeasible solution**, while $(500, 0, 500, 700, 0)$ is a **basic feasible solution**.

It is not difficult to see that when we are standing at a corner point, it is formed by the intersection of n equations, corresponding to some n constraints. At this point, the slack variable corresponding to these constraint equations is 0.

In a basic solution, the n variables that are fixed at 0 are called the **non-basic variables**, while the remaining m variables are non-zero, and are called the **basic variables**. Sometimes, a basic variable will also have 0 value (this is a degenerate case; in our example, it may occur when three lines are passing through one point).

The table below lists the basic solutions for our Product Mix problem. The *basic feasible solutions* are listed in green, while the *basic infeasible solutions* are in red.

Corner point	Feasible ?	Defining eqns	Basic solution	non-basic variables
(0, 0)	YES	$x_1 = 0$ $x_2 = 0$	(0, 0, 1500, 1200, 500)	x_1, x_2
(500, 0)	YES	$x_1 = 500$ $x_2 = 0$	(500, 0, 500, 700, 0)	x_2, x_5
(500, 500)	YES	$x_1 = 500$ $2x_1 + x_2 = 1500$	(500, 500, 0, 200, 0)	x_5, x_3
(300, 900)	YES	$x_1 + x_2 = 1200$ $2x_1 + x_2 = 1500$	(300, 900, 0, 0, 200)	x_3, x_4
(0, 1200)	YES	$x_1 = 0$ $x_1 + x_2 = 1200$	(0, 1200, 300, 0, 500)	x_4, x_1
(750, 0)	NO	$x_2 = 0$ $2x_1 + x_2 = 1500$	(750, 0, 0, 450, -250)	x_2, x_3
(1200, 0)	NO	$x_2 = 0$ $x_1 + x_2 = 1200$	(1200, 0, -900, 0, -700)	x_2, x_4
(500, 700)	NO	$x_1 + x_2 = 1200$ $x_1 = 500$	(500, 700, -200, 0, 0)	x_4, x_5
(0, 1500)	NO	$x_1 = 0$ $2x_1 + x_2 = 1500$	(0, 1500, 0, -300, 500)	x_1, x_3

There are some points to note in this table:

1. Notice that I listed the feasible solutions in a sequence such that each solution is adjacent to the ones above/below it (look at the graph in Figure 6). Now look at the non-basic variables (last column) – in each subsequent row, one variable is dropped, and one new one enters; e.g. row 1 has (x_1, x_2); in row 2, x_1 is dropped and x_5 enters... and so on.

Why is this? Notice that each corner point is the intersection of two lines; in the adjacent corner point, one of these lines is still a defining line, while the other one is replaced by a different one. The slack variable corresponding to the defining line is zero (why?). So the defining lines correspond to the non-basic variables!

2. Notice that each basic infeasible solution has at least one variable with negative value.

We are ready to see how Simplex method works – except for one detail. As we shall visit each basic feasible solution, we would like to keep track of the corresponding value of the objective function. This will help us to decide (i) which adjacent basic feasible solution we want to go to, and (ii) when to stop searching.

To do so, we will write the objective as a part of our equation: $Z = 15x_1 + 10x_2$ will be re-written as: $Z - 15x_1 - 10x_2 = 0$.

So the entire problem is written in standard form as:

maximize Z

subject to:

$$\begin{array}{rclcl}
 Z & -15x_1 & -10x_2 & & = 0 & [0-0] \\
 & 2x_1 & + x_2 & + x_3 & = 1500 & [0-1] \\
 & x_1 & + x_2 & & + x_4 & = 1200 & [0-2] \\
 & x_1 & & & & + x_5 & = 500 & [0-3] \\
 & x_i \geq 0 & \text{for all } i.
 \end{array}$$

I will follow this convention for equation numbers: [step number – equation number].

We want to search only among the basic feasible solutions. To do so, we need to ensure that (a) we start at a feasible solution, and (b) we only consider those adjacent corner points that are feasible.

Step 1. Initial solution

There are some clever techniques to select good starting points. However, for our purpose, we are happy to note that in standard form, the point $(x_1, x_2) = (0, 0)$ is certainly a corner point and is feasible; hence we start with $\{x_1, x_2\}$ as the *non-basic variables*, and the slack variables $\{x_3, x_4, x_5\}$ as the *basic variables*.

Let's write these equations:

$$\begin{array}{rclcl}
 2x_1 & + x_2 & + x_3 & & = 1500 \\
 x_1 & + x_2 & & + x_4 & = 1200 \\
 x_1 & & & & + x_5 & = 500
 \end{array}$$

I have deliberately written the non-basic variables in green; since they are all set to 0, it is very convenient to read out the values of the basic variables directly: $x_3 = 1500$, $x_4 = 1200$, $x_5 = 500$.

You will see that Simplex method maintains this property at each step: *basic variables always have a coefficient of +1, and there is exactly one basic variable in each equation*. How nice!

Step 2. Iteration Step

At each step, we will move from the current basic feasible solution (corner point feasible solution) to a **better adjacent** basic feasible solution.

As discussed earlier, this means we need to decide (a) which non-basic variable must now become non-zero (this variable will be called the **entering variable**), and (b) from the current set of basic variables, one must be dropped, i.e., forced to be equal to 0 (hence it is called the **leaving variable**).

(a) How to decide the entering variable?

Here, we shall use a greedy method. We know that all non-basic variables are 0; let us write the objective function in terms of the non-basic variables. One of these will enter

the basic set – thus it will now become non-zero; we select that non-basic variable which will increase our objective function at the *fastest rate*.

Initially, our objective function looks like: $Z = 15x_1 + 10x_2$. The non-basic variables are x_1 and x_2 , and a unit change in x_1 gives us a better rate of increase in Z (15 units) compared to 10 units per unit increase in x_2 . So we select x_1 as the entering variable.

(b) How to determine the leaving variable?

We have decided that x_1 is the entering variable. It was one of the non-basic variables, and will now join the set of basic variables. It's value can be increased from 0 to some positive value.

At the same time, we must now force one of the current basic variables to assume a 0 value, so as to keep the total number of basic variables equal to the number of constraints. In this case, we shall have no choice – our purpose is to allow the entering variable to increase in value, while keeping the other non-basic variables at 0. This will increase the value of the objective function. As we keep increasing the objective, at some stage, one of the basic variables will become zero, beyond this point, this basic variable will become negative and the solution becomes infeasible. Thus, the first basic variable that becomes zero as we increase the value of the entering variable is the only choice for the leaving variable.

Let us see how this works out in our Product Mix problem. We know that x_1 is the entering variable, and currently the basic variables are x_3 , x_4 , and x_5 . For each of these three, we must now determine how much we can increase the value of x_1 if it is the leaving variable. This analysis is summarized in the table below.

Basic variable	Equation	Upper bound for x_1	
x_3	$x_3 = 1500 - 2x_1 - x_2$	$x_2 = 0$, so $x_1 \leq 1500/2$; UB = 750	
x_4	$x_4 = 1200 - x_1 - x_2$	$x_2 = 0$, so $x_1 \leq 1200$; UB = 1200	
x_5	$x_5 = 500 - x_1$	$x_1 \leq 500$	← minimum

From the above, we can conclude that it is impossible to increase x_1 beyond 500, since after that constraint [0-3] will be violated. Thus this is the limiting constraint, and the boundary equation corresponding to this constraint must leave the set of basic variables (in other words, once x_1 is increased to 500, the constraint $x_1 + x_5 = 500$ will be used to define the next corner point, which also implies that $x_5 = 0$)

Once we identify the entering and leaving variables, we must now compute the new values of the other basic variables (obviously the non-basic variables, including the new one, x_5 , are zero; also, $x_1 = 500$). To do so, we will use a method identical to that in Gaussian elimination: multiply a row by a suitable non-zero constant, and/or add a multiple of one equation to another. Since all our constraints are equalities, doing these operations gives a totally equivalent set of equations.

The goal of these operations is to once again have our equations in a form such that the coefficient of each basic variable is +1, and each basic variable appears in exactly one equation.

Let's first re-write the original constraint equations, but this time marking the new set of basic variables in red color:

$$\begin{array}{rcll}
 Z & -15 \mathbf{x_1} & -10x_2 & = 0 & [0-0] \\
 & 2 \mathbf{x_1} & + x_2 & + \mathbf{x_3} & = 1500 & [0-1] \\
 & \mathbf{x_1} & + x_2 & & + \mathbf{x_4} & = 1200 & [0-2] \\
 & \mathbf{x_1} & & & & + x_5 & = 500 & [0-3]
 \end{array}$$

Now x_1 has replaced x_5 as the basic variable in [0-3]; here, the coefficient of x_1 is +1, so this equation is already in the form we wanted.

We want to eliminate x_1 from all other equations. To do so, we shall replace [0-2] by ([0-2] – [0-3]); likewise, we will replace [0-1] by ([0-1] – 2x [0-3]). Finally, we also eliminate x_1 from the objective equation, by replacing [0-0] with ([0-0] + 15x [0-3]).

This gives us the new set of equations:

$$\begin{array}{rcll}
 Z & & -10x_2 & & +15 \mathbf{x_5} & = 7500 & [1-0] \\
 & & x_2 & + \mathbf{x_3} & & - 2x_5 & = 500 & [1-1] \\
 & & x_2 & & + \mathbf{x_4} & - x_5 & = 700 & [1-2] \\
 & \mathbf{x_1} & & & & + x_5 & = 500 & [1-3]
 \end{array}$$

As you can see, we can directly read the values of all the variables from here; this basic feasible solution is (500, 0, 500, 700, 0). You can compare this with the values we had computed earlier (look at the table in page 7). Also note what happened to the objective value after the row operations.

Step 3. The stopping rule

How do we know that we have found the optimum solution? Let us re-write the objective equation [1-0] as follows:

$$Z = 7500 + 10x_2 - 15x_5$$

It is written in terms of the non-basic variables. We can immediately see that if the value of the non-basic variable x_2 is increased (it is currently 0), then the objective will increase (why?). On the other hand, if the value of the non-basic variable x_5 increases, it will decrease the objective value (why?).

Thus we must repeat Step 2 again, with the set of equations [1-0] through [1-3]. Again, the choice of the entering variable is clear: only x_2 can enter the basic set.

To determine the leaving variable, we must examine how much we can increase x_2 by eliminating one of the basic variables $\{x_1, x_3, x_4\}$ from the basic set. The table below summarizes this analysis:

Basic variable	Equation	Upper bound for x_2	
x_3	$x_3 = 500 - x_2 + 2x_5$	$x_2 \leq 500$	← minimum
x_4	$x_4 = 700 - x_2 + x_5$	$x_2 \leq 700$	
x_1	$x_1 = 500 - x_5$	no limit on x_2	

Thus x_3 is the leaving variable. And again, we can perform our row operations to put our constraints back into the nice form. The operations are performed in the following sequence: $[2-1] = [1-1]$ (since coefficient of x_2 is already +1 here); $[2-0] = [1-0] + 10x[2-1]$; $[2-2] = [1-2] - [2-1]$; $[2-3] = [1-3]$.

$$\begin{array}{rclcl}
 Z & & +10x_3 & -5x_5 & = 12,500 & [2-0] \\
 & \mathbf{x_2} & +x_3 & -2x_5 & = 500 & [2-1] \\
 & & -x_3 & +\mathbf{x_4} & +x_5 & = 200 & [2-2] \\
 & \mathbf{x_1} & & & +x_5 & = 500 & [2-3]
 \end{array}$$

Once again, from this nice form, we can directly read off the values of the corner solution as (500, 500, 0, 200, 0), with an objective value of 12500.

The following points are important:

- (i) You can check from figure 6 that we have moved to an adjacent basic feasible solution
- (ii) we have increased the objective value
- (iii) we are not yet at optimum, since there is at least one variable with a negative coefficient in the objective equation – in fact, there is only one, x_5 , and therefore we must use it as the next entering variable.

So x_5 will now enter as a basic variable, and therefore one of the current three basic variables, x_1 , x_2 , and x_4 must leave. Once again, we do the tests to find which of these three gives the tightest limit on our attempt to increase x_5 from 0.

Basic variable	Equation	Upper bound for x_5	
x_2	$x_2 = 500 + 2x_5 - x_3$	no limit on x_2	← minimum
x_4	$x_4 = 200 + x_3 - x_5$	$x_5 \leq 200$	
x_1	$x_1 = 500 - x_5$	$x_5 \leq 500$	

In this iteration, x_5 enters, and x_4 is the leaving variable; we can increase the value of x_5 by up to 200 units. Let us do the row operations once again on equations [2-0] through [2-3] to eliminate x_5 from all except one: $[3-0] = [2-0] + 5x[2-2]$; $[3-1] = [2-1] + 2x[2-2]$; $[3-2] = [2-2]$; $[3-3] = [2-3] - [2-2]$. The new set of equations is (with basic variables shown in bold red):

Z		+5 x ₃	+ 5x ₄	= 13,500	[3-0]	
	x₂	- x ₃	+ 2x ₄	= 900	[3-1]	
		- x ₃	+ x ₄	+ x₅	= 200	[3-2]
	x₁	+ x ₃	- x ₄	= 300	[3-3]	

Once again, one can directly read off the values of the variables directly from these equations as: (300, 900, 0, 0, 200). You can go back to our table on page 7 to verify that this is indeed a corner point feasible solution, and it corresponds to the objective value of 13,500.

At this stage, if we look at the objective equation, [3-3], we see that it is of the form: $Z = 13,500 - 5x_3 - 5x_4$. Now our two non-basic variables are x_3 and x_4 . If we pick either of these to enter as a basic variable, its value will increase from 0. But this can only decrease the value of Z , since the coefficients of each, x_3 and x_4 , is negative. This is precisely the termination condition of Simplex! In other words, we are now standing on a corner point feasible solution that is better than any of its neighbors, and therefore is the globally optimum solution.

Other important points

1. *Higher dimensions.* Notice that the Simplex method works equally well for problems of higher dimensions. The main reason is the insight that at each step, we maintain the constraint equations in the nice form, with only one basic variable per equation. Therefore, in the next iteration, there is exactly one entering variable, and exactly one leaving variable, and so, the upper bound on the entering variable can be determined easily by solving a linear equation of one variable.

The geometric insight here is that when we move to an adjacent basic solution, we must move along a line-segment (which is an edge of the feasible set in n -dimensional space). Line segments are one-dimensional, so computing upper bound is easy.

2. *Many candidates for entering variable.* What happens if there are two or more variables in the objective function with equal, positive coefficients? For example, if we have, at some step of the iterations, $Z = 200 + 10x_1 + x_2 + 10x_3$. Both x_1 and x_3 give the same rate of increase in the objective. In such cases, we randomly select either of these and proceed with the method.

3. *Many candidates for leaving variable.* What happens if there are two or more variables that give the same upper bound on an entering variable? When this occurs, we say that there is some degeneracy in the constraint set. This case is much more difficult to handle. Fortunately, this rarely occurs in real life problems, so for practical purposes, we are free to ignore it in an introductory course.

4. *No candidate for leaving variable.* Consider the case when we have an entering variable, but there is no leaving variable. In other words, each constraint allows us to increase the entering variable candidate to be increased to infinity. This occurs if our constraint set is unbounded, and therefore it appears that Z can be increased to infinity. Of course, it is easy to construct examples of this kind (think of an unbounded feasible set as a polyhedral cone with the tip at the origin, and extending outwards to infinity in the positive octant of our Cartesian frame).

For LP models of real life problems, when this occurs, it is almost guaranteed that you forgot to enter a constraint, or mistyped a constraint.

5. *Minimization problems.* We close this discussion with the following comment. Up until now, all my discussion has been assuming that we have an objective function that needs to be maximized. This is true in Product Mix type of problems, where we want to maximize the profit. But how about problems where the objective is to minimize something; for example, in our transportation model, the objective was to minimize the cost. Will Simplex work in this case? The answer is yes. The reason is that if we are trying to minimize some function, Z , it is the same thing as trying to maximize the function, $-Z$. So when faced with a minimization problem, we can immediately convert it to a maximization problem by merely multiplying the objective function by -1 , and happily use the same method to the resulting LP.

References:

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Operations Research, F. S. Hillier and G. J. Lieberman, Holden-Day