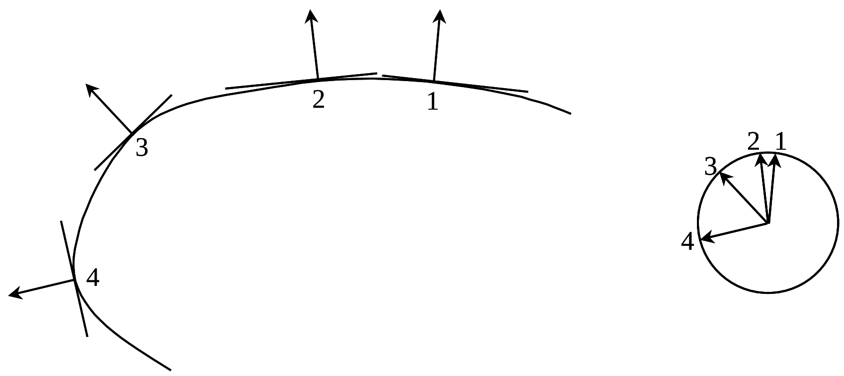
Additional notes on curvature

Much of our current concepts of curvature are based on the works of Euler and Gauss. Consider a smooth planar curve as shown in the figure below, with tangents defined on each point (some of the tangent lines are shown in the figure. We can then derive the unit normal vector at each point fairly easily, since $\boldsymbol{T.N} = 0$.



Corresponding to each point p on the curve, we draw a unit normal. If we collect all the unit normal vectors with their tails at a common point, we get a set of vectors, each with its head on the unit circle as shown in the right side in figure 1. The mapping of each point p to a point on the unit circle, $\gamma(p)$, is called the ***Gauss map*** of the curve. In general, Gauss maps can be drawn for arbitrary 2- or 3-D shapes in the same way, and have many useful applications in differential as well as computational geometry.
The curvature $\kappa(p)$ of the curve at point p is defined as the rate at which the vector $\gamma(p)$ is rotating as you move along the curve across point p. Let us take a small piece of the curve of length l, in the neighborhood of point p. Let the angle between the gauss map of the end points if this (small) segment of the curve be ϕ. Then the average curvature of this piece of the curve = ϕ/l. As we make l smaller, in the limiting case, the value of l will approach a constant number for the point p, denoted κ.
In other words, the derivative of the Gauss map at the point p is defined as its curvature, κ.

Look at the curve above, and convince yourself that the curvature at the point 4 will be higher than the curvature at point 2 (how fast is the head of the Gauss map moving ?).
Similarly, convince yourself, using symmetry, that this rate of change will be constant for each point p, if the curve is a circle. In fact, for a circle of radius r, the curvature will be exactly = $1/r$.

This idea of curvature can easily be extended to 3D curves – imagine a very small neigborhood of the point p, containing just one point on each side of p. These three points define a plane, and in this plane we can draw a circle passing through these three points. This circle is called the osculating circle, and its curvature defines the curvature of the curve at point p, in the limiting case.

Now let us look at curvature of surfaces. The idea of intersecting plane with the surface, and looking at the curvature of the resulting intersection curve was proposed by Euler. Suppose we have a smooth surface, S, with a point p on it. A line l, thorough p, perpendicular to the tangent plane at p is called the normal line. To describe the way that the surface curves at point p, we can look at the curvature of the planar curves formed by intersecting the surface S with planes containing the line l. Such planes are called normal sections of the surface at point p. Euler proved the remarkable result that the curvature κ at point p is given by the formula:

$$\kappa = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta, \text{ where}$$

κ_1 and κ_2 are the maximum and minimum values of κ, and θ is the angle between the plane corresponding to κ_1 and that for κ. In particular, the planes corresponding to κ_1 and κ_2 are perpendicular to each other!

This result is amazing because it gives us to define the curvature of a surface at any point by means of two numbers. Even more amazingly, it can even be proved that the curvatures of non-normal sections can also be defined in terms of just the two principal curvature values.

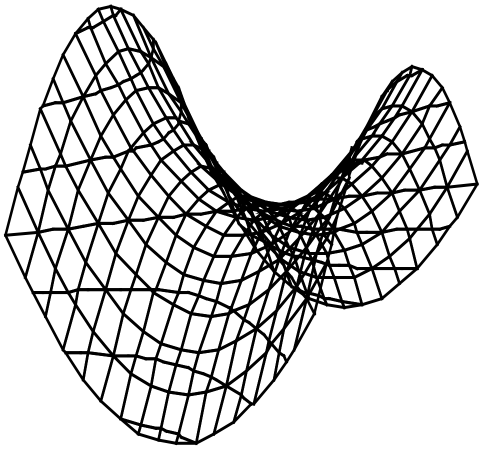
However, there is a small ambiguity in the value of the curvature as defined above, since at any given point, the normal may have two opposing directions (inward pointing and outward pointing). The curvature calculated using one normal will be the negative of that computed using the reverse. Modern definitions of curvature were formulated through the brilliant work of Gauss. For any point on the surface, Gauss defined the Gaussian map just as he did for curves, except that the normal vectors will have heads on the surface of a unit sphere.

For each point P a surface Σ, the Gaussian map maps to a point $\Gamma(P)$ on the surface of a unit sphere such that the (arbitrarily specified) outward normal of Σ at P is along the direction of $\Gamma(P)$. Gauss defined the integral curvature of a region of Σ to be the area of its image under this mapping. Further, the measure of curvature $K(P)$, at a given point P, is the limit of the ratio of the integral curvature and the area of a small neighborhood of P. The most amazing result of Gauss was that he then showed that his definition of curvature is related to Euler's results by the simple formula:

$$K(P) = \kappa_1 \kappa_2$$

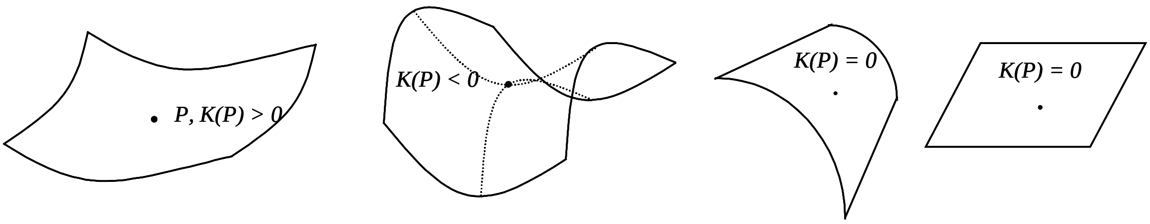
Notice now, that if change our choice of 'outward' normal to the reverse of the direction we (arbitrarily) chose earlier, then both κ_1 and κ_2 change sign, so $K(P)$ is invariant. In other words, the sign of $K(P)$ does not depend on which side of the surface we define as the 'outside'. So what does it depend on ?

For a convex surface, every normal curve on the surface has a positive curvature, and so $K(P)$ is positive. However, when we have a surface with a saddle point (for example the surface in the figure below, corresponding to $z = x^2 - y^2$), $K(P)$ at $(0,0)$ is negative, since κ_1 and κ_2 have opposing signs.



Although it is not directly related to CAD or machining of surfaces, here is another amazing property of $K(P)$ – it is a property of the ***intrinsic geometry*** of the surface. This requires some more reading, but roughly speaking, if you bend a flexible surface such that it does not stretch or compress in any region, then its intrinsic geometry is said to remain unchanged. For example, if you take a sheet of paper, and bend it into a cylinder, its intrinsic geometry is unchanged. Gauss showed that $K(P)$ of a surface in ***invariant*** – that is, by bending the surface, we may change κ_1 and κ_2 at any point, but the product of κ_1 and κ_2 is always a constant unless the surface is stretched/compressed! For example, a flat sheet of paper has $\kappa_1 = \kappa_2 = 0$ everywhere. By bending it into a cylinder of radius r, the curvature κ_1 at any point becomes $1/r$, but $\kappa_2 = 0$, and so $K(P) = 0$. Not only is this result quite incredible, it is also useful in engineering. For example, given a surface, how can we find if it is developable (that is, we can bend it to a flat shape) ? From the results of Gauss, if $K(P) \neq 0$ at any point, then we know that the surface is not developable.

As a consequence of the work of Gauss, we can say that each point on a surface belongs to one out of three types, depending on whether K(P) is <, >, or = 0. The third case is sometimes sub-classified into planar and cone-surface points, as shown in the figure below.



Source:

Geometry: Plane and Fancy, David A Singer, Springer-Verlag, New York, 1998